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Subspace identification of (AR)ARMAX, Box-Jenkins, and generalized model structures

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Abstract: An algorithm is proposed for the subspace identification of a generalized discrete linear time invariant model structure, composing of a deterministic and a stochastic subsystem that are only partially coupled. In a first step, the impulse response of the deterministic subsystem and the correlation function of the stochastic subsystem are estimated in a statistically consistent way. The second step consists of system realization, where a specific choice of the state-space basis, that reveals the deterministic and the stochastic dynamics and their coupling, is imposed. A simulation example illustrates the identification of a system that is excited by a known input and by colored noise.

1. INTRODUCTION

The identification of a discrete linear time-invariant (LTI) system involving both deterministic dynamics (due to observed inputs) and stochastic dynamics (due to unobserved inputs and output noise) starts with the selection of a proper model structure. Following the terminology of Ljung [1999], a discrimination can be made between the following model structures, depending on the degree of coupling between the deterministic and the stochastic subsystems:

ARMAX the deterministic and stochastic subsystems are fully coupled, i.e., they share all their poles;
Box-Jenkins the deterministic and stochastic subsystems are not considered as coupled;
ARARMAX the deterministic subsystem shares all of its poles with the stochastic subsystem, but not vice versa;
Generalized the deterministic and stochastic subsystems share only part of their poles.

In state-space form, the ARMAX model structure is the one that is predominantly considered in the subspace identification literature, see, e.g., the surveys of Viberg [1995], Van Overschee and De Moor [1996], and Bauer [2005]. When the deterministic and stochastic dynamics are uncoupled, the orthogonal decomposition algorithm of Verhaegen [1993] and Picci and Katayama [1996] can be used to identify a Box-Jenkins model structure, viz., a completely decoupled state-space model. Chiuso and Picci [2004] derived that in case a joint model is identified, while in reality the deterministic and stochastic dynamics are uncoupled, the resulting ARMAX model is badly overparameterized and suffers from worse numerical conditioning than a separately parameterized model.

In this paper, the subspace identification of the generalized model structure, comprising ARMAX, Box-Jenkins, and ARARMAX models as special cases, is treated. A generalized state-space model is presented in section 2. The proposed identification algorithm consists of two steps. In a first step, the impulse response of the deterministic subsystem and the correlation function of the stochastic subsystem are estimated in a statistically consistent way, as discussed in sections 3 and 4, respectively. The second step consists of the realization of the generalized state-space model, where a specific choice of the state-space basis, that reveals the deterministic and stochastic dynamics and their coupling, is imposed (see section 5). In section 6, a numerical simulation is presented, where an ARARMAX model structure is identified.

2. A GENERALIZED STATE-SPACE MODEL

A state-space model that describes a generalized combined deterministic-stochastic discrete LTI system is [1]

$$\underbrace{\begin{bmatrix} \boldsymbol{x}^1_{k+1} \\ \boldsymbol{x}^2_{k+1} \\ \boldsymbol{x}^3_{k+1} \end{bmatrix}}_{\boldsymbol{x}_{k+1}} = \underbrace{\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_{33} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} \boldsymbol{x}^1_{k} \\ \boldsymbol{x}^2_{k} \\ \boldsymbol{x}^3_{k} \end{bmatrix}}_{\boldsymbol{x}_{k}} + \underbrace{\begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \\ \boldsymbol{0} \end{bmatrix}}_{\boldsymbol{B}} \boldsymbol{u}_k + \underbrace{\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{K}^2_k \\ \boldsymbol{K}^3_k \end{bmatrix}}_{\boldsymbol{K}_k} \boldsymbol{e}_k \tag{1}$$

$$\boldsymbol{y}_k = \underbrace{[\boldsymbol{C}_1 \ \boldsymbol{C}_2 \ \boldsymbol{C}_3]}_{\boldsymbol{C}} \begin{bmatrix} \boldsymbol{x}^1_{k} \\ \boldsymbol{x}^2_{k} \\ \boldsymbol{x}^3_{k} \end{bmatrix} + \boldsymbol{D}\boldsymbol{u}_k + \boldsymbol{e}_k, \tag{2}$$

where $\boldsymbol{y}_k \in \mathbb{R}^{n_y}$ is the vector with observed outputs, $\boldsymbol{u}_k \in \mathbb{R}^{n_u}$ the vector with observed inputs, $\boldsymbol{x}^1_k \in \mathbb{R}^{n_1}$ the

[1] By replacing the non-stationary Kalman filter by a stationary one and applying a z-transform and Cramer's rule, it is formally shown that each element of the obtained common denominator transfer function model satisfies the generalized model structure definition of Ljung [1999].

part of the state vector $\boldsymbol{x_k} \in \mathbb{R}^n$ that describes the purely deterministic dynamics, $\boldsymbol{x_k^2} \in \mathbb{R}^{n_2}$ describes the common dynamics and $\boldsymbol{x_k^3} \in \mathbb{R}^{n_3}$ describes the purely stochastic dynamics. $\boldsymbol{K_k} \in \mathbb{R}^{n \times n_y}$ is a non-stationary Kalman filter corresponding to the innovation vector $\boldsymbol{e_k} \in \mathbb{R}^{n_y}$.

The generalized combined deterministic-stochastic description (1-2) can be formally decoupled in a deterministic subsystem

$$\underbrace{\begin{bmatrix} \boldsymbol{x_{k+1}^1} \\ \boldsymbol{x_{k+1}^{2d}} \end{bmatrix}}_{\boldsymbol{x_{k+1}^d}} = \underbrace{\begin{bmatrix} \boldsymbol{A_{11}} & \boldsymbol{0} \\ \boldsymbol{A_{21}} & \boldsymbol{A_{22}} \end{bmatrix}}_{\boldsymbol{A_d}} \underbrace{\begin{bmatrix} \boldsymbol{x_k^1} \\ \boldsymbol{x_k^{2d}} \end{bmatrix}}_{\boldsymbol{x_k^d}} + \underbrace{\begin{bmatrix} \boldsymbol{B_1} \\ \boldsymbol{B_2} \end{bmatrix}}_{\boldsymbol{B_d}} \boldsymbol{u_k} \tag{3}$$

$$\boldsymbol{y_k^d} = \underbrace{[\boldsymbol{C_1} \;\; \boldsymbol{C_2}]}_{\boldsymbol{C_d}} \boldsymbol{x_k^d} + \boldsymbol{D u_k}, \tag{4}$$

and a stochastic subsystem

$$\underbrace{\begin{bmatrix} \boldsymbol{x_{k+1}^{2s}} \\ \boldsymbol{x_{k+1}^3} \end{bmatrix}}_{\boldsymbol{x_{k+1}^s}} = \underbrace{\begin{bmatrix} \boldsymbol{A_{22}} & \boldsymbol{A_{23}} \\ \boldsymbol{0} & \boldsymbol{A_{33}} \end{bmatrix}}_{\boldsymbol{A_s}} \underbrace{\begin{bmatrix} \boldsymbol{x_k^{2s}} \\ \boldsymbol{x_k^3} \end{bmatrix}}_{\boldsymbol{x_k^s}} + \underbrace{\begin{bmatrix} \boldsymbol{K_k^2} \\ \boldsymbol{K_k^3} \end{bmatrix}}_{\boldsymbol{K_k^s}} \boldsymbol{e_k} \tag{5}$$

$$\boldsymbol{y_k^s} = \underbrace{[\boldsymbol{C_2} \;\; \boldsymbol{C_3}]}_{\boldsymbol{C_s}} \boldsymbol{x_k^s} + \boldsymbol{e_k}, \tag{6}$$

where $\boldsymbol{y_k} = \boldsymbol{y_k^d} + \boldsymbol{y_k^s}$ and $\boldsymbol{x_k^2} = \boldsymbol{x_k^{2d}} + \boldsymbol{x_k^{2s}}$.

3. OBTAINING THE IMPULSE RESPONSE OF THE DETERMINISTIC SUBSYSTEM

This section deals with the estimation of the first $2\imath$ impulse response matrices $\boldsymbol{H_k}$, $k = 0, \ldots, 2\imath - 1$ of the deterministic subsystem (3-4).

3.1 Notations

Denote $\boldsymbol{\mathcal{H}_{0|2\imath-1}}$ as the matrix containing the stacked impulse response matrices, i.e.,

$$\boldsymbol{\mathcal{H}_{0|2\imath-1}} \triangleq \left[\boldsymbol{H_0^T} \;\; \boldsymbol{H_1^T} \;\; \ldots \;\; \boldsymbol{H_{2\imath-1}^T}\right].$$

Suppose an input-output trajectory w_N, generated by the system (1-2), is observed:

$$w_N \triangleq \{(\boldsymbol{u_0}, \boldsymbol{y_0}), (\boldsymbol{u_1}, \boldsymbol{y_1}), \ldots, (\boldsymbol{u_{N-1}}, \boldsymbol{y_{N-1}})\}.$$

A block Hankel matrix of a subsequence of $(\boldsymbol{q_k}), k = 0, \ldots, N-1$ is denoted as

$$\boldsymbol{Q_{k_1|k_2}} \triangleq \begin{bmatrix} \boldsymbol{q_{k_1}} & \boldsymbol{q_{k_1+1}} & \cdots & \boldsymbol{q_{k_1+\jmath-1}} \\ \boldsymbol{q_{k_1+1}} & \boldsymbol{q_{k_1+2}} & \cdots & \boldsymbol{q_{k_1+\jmath}} \\ \vdots & \vdots & \ldots & \vdots \\ \boldsymbol{q_{k_2}} & \boldsymbol{q_{k_2+1}} & \cdots & \boldsymbol{q_{k_2+\jmath-1}} \end{bmatrix},$$

where $0 \le k_1 < k_2 < 2(\imath+\ell)-1$ and $\jmath = N-2(\imath+\ell)-1$ with ℓ strictly larger than the system's time lag (i.e., the system order plus the delay, see Willems et al. [2005] for a formal definition), and its row span is denoted as $\mathscr{Q}_{k_1|k_2}$. So, $\boldsymbol{Y_{k_1|k_2}}$ is a block Hankel matrix of outputs with row span $\mathscr{Y}_{k_1|k_2}$, $\boldsymbol{U_{k_1|k_2}}$ is a block Hankel matrix of inputs with row span $\mathscr{U}_{k_1|k_2}$, $\boldsymbol{E_{k_1|k_2}}$ is a block Hankel matrix of innovations with row span $\mathscr{E}_{k_1|k_2}$, etc. With these notations, it follows from (3-6), that

$$\begin{aligned}\boldsymbol{Y_{k_1|k_2}} = \boldsymbol{\mathcal{O}_{k_2-k_1+1}^d X_{k_1|k_1}^d} + \boldsymbol{\mathcal{F}_{k_2-k_1+1}^d U_{k_1|k_2}} + \\ \boldsymbol{\mathcal{O}_{k_2-k_1+1}^s X_{k_1|k_1}^s} + \boldsymbol{\mathcal{F}_{k_2-k_1+1}^s E_{k_1|k_2}},\end{aligned} \tag{7}$$

where $\boldsymbol{\mathcal{O}_{k_2-k_1+1}^d}$ and $\boldsymbol{\mathcal{O}_{k_2-k_1+1}^s}$ are the observability matrices of the deterministic and stochastic subsystems, respectively,

$$\boldsymbol{\mathcal{O}_{k_2-k_1+1}^d} \triangleq \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C_d A_d} \\ \vdots \\ \boldsymbol{C_d A_d^{k_2-k_1}} \end{bmatrix}, \quad \boldsymbol{\mathcal{O}_{k_2-k_1+1}^s} \triangleq \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C_s A_s} \\ \vdots \\ \boldsymbol{C_s A_s^{k_2-k_1}} \end{bmatrix}, \tag{8}$$

and $\boldsymbol{\mathcal{F}_{k_2-k_1+1}^d}$ and $\boldsymbol{\mathcal{F}_{k_2-k_1+1}^s}$ are defined as

$$\boldsymbol{\mathcal{F}_{k_2-k_1+1}^d} \triangleq \begin{bmatrix} \boldsymbol{D} & \boldsymbol{0} & \ldots & \boldsymbol{0} \\ \boldsymbol{C_d B} & \boldsymbol{D} & \ldots & \boldsymbol{0} \\ \vdots & \vdots & \ldots & \vdots \\ \boldsymbol{C_d A_d^{k_2-k_1-1} B} & \boldsymbol{C_d A_d^{k_2-k_1-2} B} & \ldots & \boldsymbol{D} \end{bmatrix}$$

$$\boldsymbol{\mathcal{F}_{k_2-k_1+1}^s} \triangleq \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \ldots & \boldsymbol{0} \\ \boldsymbol{C_s K_{k_1}^s} & \boldsymbol{I} & \ldots & \boldsymbol{0} \\ \vdots & \vdots & \ldots & \vdots \\ \boldsymbol{C_s A_s^{k_2-k_1-1} K_{k_2-1}^s} & \boldsymbol{C_s A_s^{k_2-k_1-2} K_{k_2-2}^s} & \ldots & \boldsymbol{I} \end{bmatrix}.$$

3.2 A strongly consistent estimator

As shown in Markovsky et al. [2005], one has that, under the assumptions that the deterministic subsystem is controllable and $(\boldsymbol{u_k})$ is persistently exciting (Willems et al. [2005]) of order $\ge 2\imath + \ell + n_1 + n_2$,

$$\begin{bmatrix} \boldsymbol{U_{0|2\imath+\ell-1}} \\ \boldsymbol{Y_{0|2\imath+\ell-1}^d} \end{bmatrix} \boldsymbol{\tilde{G}} = \begin{bmatrix} \boldsymbol{\mathcal{O}} \\ \boldsymbol{\mathcal{H}_{0|2\imath-1}} \end{bmatrix},$$

where

$$\boldsymbol{\mathcal{O}} = \left[\boldsymbol{0_{n_u \times n_u \ell}} \;\; \boldsymbol{I_{n_u}} \;\; \boldsymbol{0_{n_u \times (2\imath-1)n_u + \ell n_y}}\right]^T.$$

This equation can be solved for $\boldsymbol{\tilde{G}}$ from the first $(2\imath + \ell)n_u + \ell n_y$ rows, after which $\boldsymbol{\mathcal{H}_{0|2\imath-1}}$ is calculated exactly from the last $2\imath n_y$ rows:

$$\boldsymbol{\mathcal{H}_{0|2\imath-1}} = \boldsymbol{Y_{\ell|2\imath+\ell-1}^d} \begin{bmatrix} \boldsymbol{U_{0|2\imath+\ell-1}} \\ \boldsymbol{Y_{0|\ell-1}^d} \end{bmatrix}^{\dagger} \boldsymbol{\mathcal{O}}, \tag{9}$$

where $\square^\dagger$ denotes the Moore-Penrose pseudo-inverse (Ben-Israel and Greville [1974]).

However, the deterministic outputs in $\boldsymbol{Y_{0|2\imath+\ell-1}^d}$ are not directly observed, and this deterministic algorithm should be changed in case combined deterministic-stochastic dynamics are present. Hereto, the following assumptions are made.

Assumption 1. The input sequence $(\boldsymbol{u_k}), k = 0, \ldots, N-1$ is observed free of noise and it is persistently exciting of order $\ge 2(\imath + \ell) + n_1 + n_2$. The deterministic system (3-4) is controllable.

Assumption 2. The stochastic outputs are uncorrelated with the observed inputs, i.e.,

$$\forall k, l : \mathcal{E}\left(\boldsymbol{y_k^s u_l^T}\right) = \boldsymbol{0},$$

where $\mathcal{E}$ denotes the expectation operator.

A possible estimate for $\boldsymbol{Y^d_{0|2\imath+\ell-1}}$ is the orthogonal projection $\boldsymbol{Y_{0|2\imath+\ell-1}}/\mathscr{U}_{0|2\imath+\ell-1}$, see, e.g., Chiuso and Picci [2004]. However, the row rank of this projection is generally smaller than $\dim(\mathscr{Y}^d_{0|2\imath+\ell-1})$, since it follows from (7) that

$$\text{rowspan}(\boldsymbol{Y^d_{0|2\imath+\ell-1}}) \subseteq \mathscr{X}^d_{0|0} \vee \mathscr{U}_{0|2\imath+\ell-1},$$

where $\vee$ denotes joint row space, and generally $\mathscr{X}^d_{0|0} \not\subseteq \mathscr{U}_{0|2\imath+\ell-1}$. As a consequence, (9) is not exact anymore when $\boldsymbol{Y^d_{0|2\imath+\ell-1}}$ is projected onto $\mathscr{U}_{0|2\imath+\ell-1}$. Therefore, the following projection is proposed.

Theorem 1. The following estimate for the impulse response sequence is statistically strongly consistent [2] for $\jmath \to \infty$ under assumptions 1 and 2:

$$\boldsymbol{\hat{\mathcal{H}}_{0|2\imath-1}} \triangleq \boldsymbol{Y_{\ell|2\imath+\ell-1}}/\mathscr{U}_{0|2(\imath+\ell)-1} \begin{bmatrix} \boldsymbol{U_{0|2\imath+\ell-1}} \\ \boldsymbol{\hat{Y}_{0|\ell-1}}/\mathscr{U}_{0|2(\imath+\ell)-1} \end{bmatrix}^{\dagger} \boldsymbol{\mathcal{O}}. \quad (10)$$

Proof. The proof follows the same lines as the proof of Theorem I in Reynders et al. [2008], where a slightly different consistent estimator is proposed. The estimator proposed here is statistically more efficient.

4. OBTAINING THE OUTPUT CORRELATION SEQUENCE OF THE STOCHASTIC SUBSYSTEM

Denote $\boldsymbol{\mathcal{L}_{1|2\imath-1}}$ as the matrix containing the stacked stochastic output correlation matrices, i.e.,

$$\boldsymbol{\mathcal{L}_{1|2\imath-1}} \triangleq \left[\boldsymbol{\Lambda_1^T}\ \boldsymbol{\Lambda_2^T}\ \dots\ \boldsymbol{\Lambda_{2\imath-1}^T}\right],$$

where $\boldsymbol{\Lambda_l}$ is defined as

$$\boldsymbol{\Lambda_l} \triangleq \mathcal{E}\left(\boldsymbol{y^s_{k+l}}\boldsymbol{y^s_l}^T\right).$$

A block vector of a vector subsequence of $(\boldsymbol{q_k}), k = 0, \dots, N-1$ is denoted as

$$\boldsymbol{q_{k_1|k_2}} \triangleq \left[\boldsymbol{q^T_{k_1}}\ \boldsymbol{q^T_{k_1+1}}\ \dots\ \boldsymbol{q^T_{k_2}}\right]^T.$$

A consistent estimator for the output correlation sequence of the stochastic subsystem is presented in the following lemma.

Lemma 2. Under assumptions 1 and 2, the following estimate for $\boldsymbol{\mathcal{L}_{1|2\imath-1}}$ is strongly consistent for $\jmath \to \infty$:

$$\boldsymbol{\hat{\mathcal{L}}_{1|2\imath-1}} \triangleq \frac{1}{\jmath}\boldsymbol{\hat{Y}^s_{1|2\imath-1}}\boldsymbol{\hat{Y}^s_{0|0}}^T.$$

where

$$\boldsymbol{\hat{Y}^s_{0|2\imath-1}} \triangleq \boldsymbol{Y_{0|2\imath-1}}\mathscr{U}^{\perp}_{0|2(\imath+\ell)-1}$$

with $\mathscr{U}^{\perp}_{0|2(\imath+\ell)-1}$ the orthogonal complement of $\mathscr{U}_{0|2(\imath+\ell)-1}$ in $\mathscr{U}_{0|2(\imath+\ell)-1} \vee \mathscr{Y}_{0|2\imath+\ell-1}$.

Proof. In theorem 1, it was shown that, under assumption 1,

$$\text{rowspan}(\boldsymbol{Y^d_{0|2\imath+\ell-1}}) \subseteq \mathscr{U}_{0|2(\imath+\ell)-1},$$

and consequently

$$\text{rowspan}(\boldsymbol{Y^d_{0|2\imath+\ell-1}}) \cap \mathscr{U}^{\perp}_{0|2(\imath+\ell)-1} = \emptyset.$$

Since under assumption 2, one has additionally that

$$\text{rowspan}(\boldsymbol{Y^s_{0|2\imath+\ell-1}}) \subseteq \mathscr{U}^{\perp}_{0|2(\imath+\ell)-1},\ \jmath \to \infty,$$

it follows that

$$\text{a.s.lim}_{\jmath\to\infty}\left(\boldsymbol{Y_{1|2\imath-1}}\mathscr{U}^{\perp}_{0|2(\imath+\ell)-1}\right)\left(\boldsymbol{Y_{0|0}}\mathscr{U}^{\perp}_{0|2(\imath+\ell)-1}\right)^T = \boldsymbol{\mathcal{L}_{1|2\imath-1}}.$$

5. REALIZATION OF THE GENERALIZED MODEL STRUCTURE

5.1 Decomposition of the impulse responses

The impulse response matrices $\boldsymbol{H_k}$ of the deterministic subsystem (3-4) can be written as a product of its system matrices:

$$\boldsymbol{H_0} = \boldsymbol{D}, \qquad \boldsymbol{H_k} = \boldsymbol{C_d A_d^{k-1} B_d} = \boldsymbol{C A^{k-1} B},\ k \geq 1.$$

From this equation, the direct transmission term $\boldsymbol{D}$ can be immediately obtained. Deterministic realization starts with gathering the other impulse response matrices in a block Hankel matrix (Ho and Kalman [1966]):

$$\boldsymbol{H_{1|\imath}} \triangleq \begin{bmatrix} \boldsymbol{H_1} & \boldsymbol{H_2} & \dots & \boldsymbol{H_\imath} \\ \boldsymbol{H_2} & \boldsymbol{H_3} & \dots & \boldsymbol{H_{\imath+1}} \\ \vdots & \vdots & \dots & \vdots \\ \boldsymbol{H_\imath} & \boldsymbol{H_{\imath+1}} & \dots & \boldsymbol{H_{2\imath-1}} \end{bmatrix}.$$

From (1-2), it follows that this matrix satisfies

$$\boldsymbol{H_{1|\imath}} = \boldsymbol{\mathcal{O}^d_\imath \mathcal{C}^d_\imath} = \boldsymbol{\mathcal{O}_\imath \mathcal{C}^D_\imath}, \quad (11)$$

where $\boldsymbol{\mathcal{O}^d_\imath}$ was defined in (8), and

$$\begin{aligned} \boldsymbol{\mathcal{C}^d_\imath} &\triangleq \left[\boldsymbol{B_d}\ \boldsymbol{A_d B_d}\ \dots\ \boldsymbol{A_d^{i-1} B_d}\right] \\ \boldsymbol{\mathcal{O}_\imath} &\triangleq \left[\boldsymbol{C}^T\ (\boldsymbol{CA})^T\ \dots\ (\boldsymbol{CA}^{i-1})^T\right]^T \\ \boldsymbol{\mathcal{C}^D_\imath} &\triangleq \left[\boldsymbol{B}\ \boldsymbol{AB}\ \dots\ \boldsymbol{A}^{i-1}\boldsymbol{B}\right]. \end{aligned} \quad (12)$$

5.2 Decomposition of the stochastic output correlations

Similarly, the output correlation matrices $\boldsymbol{\Lambda_k}$ of the stochastic subsystem (5-6) can be written as a product of its system matrices:

$$\boldsymbol{\Lambda_k} = \boldsymbol{C_s A_s^{k-1} G},\ k \geq 1,$$

where $\boldsymbol{G} \triangleq \mathcal{E}(\boldsymbol{x^s_{k+1}}\boldsymbol{y^s_k}^T)$ (Akaike [1974]). Stochastic realization starts with gathering these output correlations in a block Hankel matrix:

$$\boldsymbol{\Lambda_{1|i}} \triangleq \begin{bmatrix} \boldsymbol{\Lambda_1} & \boldsymbol{\Lambda_2} & \dots & \boldsymbol{\Lambda_\imath} \\ \boldsymbol{\Lambda_2} & \boldsymbol{\Lambda_3} & \dots & \boldsymbol{\Lambda_{\imath+1}} \\ \vdots & \vdots & \dots & \vdots \\ \boldsymbol{\Lambda_\imath} & \boldsymbol{\Lambda_{\imath+1}} & \dots & \boldsymbol{\Lambda_{2\imath-1}} \end{bmatrix},$$

[2] This means that the estimate converges to its true value 'with probability one' or 'almost surely', see, e.g., Pintelon and Schoukens [2001].

which decomposes as

$$\mathbf{\Lambda}_{1|\imath} = \mathcal{O}_\imath^s \mathcal{C}_\imath^s = \mathcal{O}_\imath \mathcal{C}_\imath^S, \tag{13}$$

where $\mathcal{O}_\imath^s$ and $\mathcal{O}_\imath$ were defined in (8) and (12), respectively, and

$$\mathcal{C}_\imath^s \triangleq \left[\boldsymbol{G}\ \boldsymbol{A}_s\boldsymbol{G}\ \ldots\ \boldsymbol{A}_s^{i-1}\boldsymbol{G}\right]$$
$$\mathcal{C}_\imath^S \triangleq \left[\begin{bmatrix}\mathbf{0}\\ \boldsymbol{G}\end{bmatrix}\ \boldsymbol{A}\begin{bmatrix}\mathbf{0}\\ \boldsymbol{G}\end{bmatrix}\ \ldots\ \boldsymbol{A}^{i-1}\begin{bmatrix}\mathbf{0}\\ \boldsymbol{G}\end{bmatrix}\right].$$

5.3 Combined decomposition

From the decompositions (11) and (13), it follows that

$$\boldsymbol{G}_{1|\imath} \triangleq \left[\boldsymbol{H}_{1|\imath}\ \mathbf{\Lambda}_{1|\imath}\right] = \mathcal{O}_\imath\left[\mathcal{C}_\imath^D\ \mathcal{C}_\imath^S\right].$$

It should be noted that, depending on the input and output units chosen, $\boldsymbol{H}_{1|\imath}$ and $\mathbf{\Lambda}_{1|\imath}$ might be of completely different orders of magnitude. Therefore, re-scaling them could be necessary, for example as

$$\boldsymbol{G}_{1|\imath}^r \triangleq \left[\boldsymbol{H}_{1|\imath}\ \frac{\|\boldsymbol{H}_{1|\imath}\|}{\|\mathbf{\Lambda}_{1|\imath}\|}\mathbf{\Lambda}_{1|\imath}\right].$$

The matrices $\mathcal{O}_\imath$, $\mathcal{C}_\imath^D$ and $\mathcal{C}_\imath^S$ can be obtained from $\boldsymbol{G}_{1|\imath}$, up to a similarity transformation of the $\boldsymbol{A}$ matrix, using reduced singular value decomposition (Zeiger and McEwen [1974], Kung [1978]):

$$\begin{aligned}
\boldsymbol{G}_{1|\imath} &\triangleq \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T \triangleq \boldsymbol{U}\boldsymbol{S}\left[\boldsymbol{V}_{\boldsymbol{D}}^T\ \boldsymbol{V}_{\boldsymbol{S}}^T\right] && (14)\\
\mathcal{O}_\imath &= \boldsymbol{U}\boldsymbol{S}^{1/2}\boldsymbol{T}^{-1} && (15)\\
\mathcal{C}_\imath^D &= \boldsymbol{T}\boldsymbol{S}^{1/2}\boldsymbol{V}_{\boldsymbol{D}}^T\\
\mathcal{C}_\imath^S &= \boldsymbol{T}\boldsymbol{S}^{1/2}\boldsymbol{V}_{\boldsymbol{S}}^T
\end{aligned}$$

where $\boldsymbol{S} \in \mathbb{R}^{n\times n}$ contains the nonzero singular values and $\boldsymbol{U} \in \mathbb{R}^{\imath n_y\times n}$ and $\boldsymbol{V} \in \mathbb{R}^{\imath n_u\times n}$ contain the corresponding singular vectors. $\boldsymbol{T} \in \mathbb{R}^{n\times n}$ is an arbitrary nonsingular matrix, that is chosen in such a way that the parametrization (1-2) is satisfied, i.e., in such a way that the last n_3 rows of $\mathcal{C}_\imath^D$ are zero while the first n_1 rows of $\mathcal{C}_\imath^S$ are zero as well.

5.4 Choice of state-space basis

A two-step procedure is followed for the construction of $\boldsymbol{T}$:

$$\boldsymbol{T} = \boldsymbol{T}_1\boldsymbol{T}_2,$$

where $\boldsymbol{T}_1$ is determined in the first step, and $\boldsymbol{T}_2$ in the second step. In the first step, we try to make the last n_3 rows of $\mathcal{C}_\imath^D$ zero. This is achieved by the following full singular value decomposition

$$\boldsymbol{S}^{1/2}\boldsymbol{V}_{\boldsymbol{D}}^T \triangleq \boldsymbol{U}^{\boldsymbol{D}}\boldsymbol{S}^{\boldsymbol{D}}\boldsymbol{V}^{\boldsymbol{D}},$$

where $\boldsymbol{S}^{\boldsymbol{D}} \in \mathbb{R}^{n\times n}$ is the diagonal matrix containing the singular values, the last n_3 of which are zero. Since consequently the last n_3 rows of $\boldsymbol{S}^{\boldsymbol{D}}\boldsymbol{V}^{\boldsymbol{D}}$ are zero, a good choice for $\boldsymbol{T}_1$ is

$$\boldsymbol{T}_1 = \boldsymbol{U}^{\boldsymbol{D}^T}.$$

In the second step, we try to make the first n_1 rows of $\mathcal{C}_\imath^S$ zero, while ensuring that the last n_3 rows of $\mathcal{C}_\imath^D$ are kept to zero. This is again achieved by a full singular value decomposition,

$$\boldsymbol{U}^{\boldsymbol{D}^T}\boldsymbol{S}^{1/2}\boldsymbol{V}_{\boldsymbol{S}}^T \triangleq \boldsymbol{U}^{\boldsymbol{S}}\boldsymbol{S}^{\boldsymbol{S}}\boldsymbol{V}^{\boldsymbol{S}},$$

where $\boldsymbol{S}^{\boldsymbol{S}} \in \mathbb{R}^{n\times n}$ is a diagonal matrix containing the singular values, that are arranged in such a way that the $n_1\times n_1$ upper left part contains the n_1 zero singular values. Since with this arrangement the first n_1 rows of $\boldsymbol{S}^{\boldsymbol{S}}\boldsymbol{V}^{\boldsymbol{S}}$ are zero, a good choice for $\boldsymbol{T}_2$ is

$$\boldsymbol{T}_2 = \left[\begin{array}{c}{\boldsymbol{U}^{\boldsymbol{S}}_{1:n_1+n_2,:}}^T\\ \hline \mathbf{0}\ \boldsymbol{I}_{n_3}\end{array}\right], \tag{16}$$

where $\boldsymbol{U}^{\boldsymbol{S}}_{1:n_1+n_2,:}$ denotes the matrix containing the first n_1+n_2 rows of $\boldsymbol{U}^{\boldsymbol{S}}$. The following lemma follows.

Lemma 3. When $n_u\imath \geq (n_1+n_2)$ and $n_y\imath \geq n$, the exact decomposition (14-16) ensures that the parametrization (1-2) is satisfied.

5.5 Obtaining the system matrices

$\boldsymbol{C}$ can be determined as the first n_y rows of $\mathcal{O}_\imath$, $\boldsymbol{B}$ as the first n_u columns of $\mathcal{C}_\imath^D$, $\boldsymbol{G}$ as the first n_y columns of $\mathcal{C}_\imath^S$, and $\boldsymbol{A}$ from the shift structure of the matrix $\mathcal{O}_\imath$ (Kung [1978]):

$$\boldsymbol{A} = \underline{\mathcal{O}_\imath}^\dagger\overline{\mathcal{O}_\imath} \tag{17}$$

where $\underline{\mathcal{O}_\imath}$ is equal to $\mathcal{O}_\imath$ without the last l rows and $\overline{\mathcal{O}_\imath}$ is equal to $\mathcal{O}_\imath$ without the first l rows. Since the parametrization (1-2) is satisfied in the above decomposition, the submatrices of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ in (1-2) are simply obtained by dividing the estimated system matrices into blocks of appropriate dimensions.

Corollary 4. From theorem 1 and lemmas 2 and 3, it follows that, since the almost sure limit and a continuous function may be interchanged, the estimates for the system matrices are strongly consistent for $\jmath \to \infty$ under the adopted assumptions.

5.6 Finite number of samples

For finite values of $\jmath$, the orders n_1, n_2 and n should be estimated first, for instance as the number of significant singular values of $\hat{\boldsymbol{H}}_{1|\imath}$, $\hat{\mathbf{\Lambda}}_{1|\imath}$, and $\hat{\boldsymbol{G}}_{1|\imath}$, respectively. Replacing $\boldsymbol{H}_{1|\imath}$ and $\mathbf{\Lambda}_{1|\imath}$ by their estimates $\hat{\boldsymbol{H}}_{1|\imath}$ and $\hat{\mathbf{\Lambda}}_{1|\imath}$, respectively, has the effect that the $n-n_1$ smallest singular values of $\boldsymbol{S}^{\boldsymbol{D}}$ and the $n-n_3$ smallest singular values of $\boldsymbol{S}^{\boldsymbol{S}}$ are nonzero for finite $\jmath$. When $\hat{\boldsymbol{H}}_{1|\imath}$ and $\hat{\mathbf{\Lambda}}_{1|\imath}$ are replaced by their best rank n_1+n_2 and n_2+n_3 estimates, respectively, before $\boldsymbol{T}$ is determined, the last n_3 rows of $\hat{\mathcal{C}}_\imath^D$ and the first n_1 rows of $\hat{\mathcal{C}}_\imath^S$ are exactly zero. This ensures that the identified $\hat{\boldsymbol{B}}$ and $\hat{\boldsymbol{G}}$ matrices have the correct block structure. For the identified $\hat{\boldsymbol{A}}$ matrix, this is only asymptotically the case, for $\jmath \to \infty$.

5.7 (AR)ARMAX and Box-Jenkins model structures

For an ARARMAX model structure, $n_1 = 0$. The algorithm for the generalized model structure can be used, when the choice of $\boldsymbol{T_2}$ is changed to

$$\boldsymbol{T_2} = \boldsymbol{I_n}.$$

For a Box-Jenkins model structure, $n_2 = 0$. In this case, the algorithm for the generalized model structure can be used without modifications. The assumptions needed here are weaker than for the subspace algorithms for Box-Jenkins identification of for instance Verhaegen [1993], Picci and Katayama [1996], and Chiuso and Picci [2001, 2004], which require that $\mathscr{X}^d_{0|0} \subseteq \mathscr{U}_{0|2\imath+\ell-1}$.

For an ARMAX model structure, $n_1 = n_3 = 0$. The algorithm for the generalized model structure can be used, when the choice of $\boldsymbol{T}$ is changed to

$$\boldsymbol{T} = \boldsymbol{I_n}.$$

In this case, the identified system model is balanced in the deterministic sense. The presented algorithm provides an alternative for existing subspace algorithms for the identification of ARMAX models, such as the CVA (Larimore [1990]), PO-MOESP (Verhaegen [1994]), and N4SID (Van Overschee and De Moor [1994]) algorithms, but its computational cost is slightly higher.

6. APPLICATION: IDENTIFICATION OF A STATE-SPACE MODEL INVOLVING COLORED SYSTEM AND MEASUREMENT NOISE

First, it is explained how the situation where a system is persistently excited by an observed input and by colored measurement noise, leads to an ARARMAX model. Then, a simulation example is given.

6.1 The model

The 'classical' (ARMAX) combined deterministic-stochastic state-space model of a discrete LTI system reads in the notation used here

$$\boldsymbol{x^2_{k+1}} = \boldsymbol{A_{22}x^2_k} + \boldsymbol{B_2u_k} + \boldsymbol{K^2_k f_k} \tag{18}$$
$$\boldsymbol{y_k} = \boldsymbol{C_2x^2_k} + \boldsymbol{Du_k} + \boldsymbol{f_k}, \tag{19}$$

where $\boldsymbol{K^2_k f_k} \in \mathbb{R}^{n_2}$ is called the system noise, and $\boldsymbol{f_k} \in \mathbb{R}^{n_y}$ the measurement noise [3]. When the system and measurement noise satisfy a filtered white noise assumption, they can be described using the forward innovation model

$$\boldsymbol{x^3_{k+1}} = \boldsymbol{A_{33}x^3_k} + \boldsymbol{K^3_k e_k}$$
$$\boldsymbol{f_k} = \boldsymbol{C_3x^3_k} + \boldsymbol{e_k}.$$

Substitution into (18-19) yields

[3] The notation $\boldsymbol{f_k}$ is used instead of $\boldsymbol{e_k}$ to emphasize the fact that $(\boldsymbol{f_k})$ is possibly nonwhite and therefore not necessarily an innovation sequence.

$$\begin{bmatrix}\boldsymbol{x^2_{k+1}}\\ \boldsymbol{x^3_{k+1}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A_{22}} & \boldsymbol{K^2_k C_3}\\ \boldsymbol{0} & \boldsymbol{A_{33}}\end{bmatrix}\begin{bmatrix}\boldsymbol{x^2_k}\\ \boldsymbol{x^3_k}\end{bmatrix} + \begin{bmatrix}\boldsymbol{B_2}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{u_k} + \begin{bmatrix}\boldsymbol{K^2_k}\\ \boldsymbol{K^3_k}\end{bmatrix}\boldsymbol{e_k} \tag{20}$$

$$\boldsymbol{y_k} = [\boldsymbol{C_2}\ \boldsymbol{C_3}]\begin{bmatrix}\boldsymbol{x^2_k}\\ \boldsymbol{x^3_k}\end{bmatrix} + \boldsymbol{Du_k} + \boldsymbol{e_k}, \tag{21}$$

which satisfies the generalized model structure (1-6) for $n_1 = 0$ and $\boldsymbol{A_{23}} = \boldsymbol{K^2_k C_3}$. Consequently, an ARARMAX model has been obtained.

6.2 Simulation example

The identification of the ARARMAX model (20-21) using the proposed subspace algorithm is illustrated for the following system

$$\boldsymbol{A_{22}} = \begin{bmatrix}0.5321 & 0.8349\\ -0.8349 & 0.5321\end{bmatrix}, \quad \boldsymbol{A_{33}} = \begin{bmatrix}-0.9615 & 0.1313\\ -0.1313 & -0.9615\end{bmatrix}$$

$$\boldsymbol{B_2} = \begin{bmatrix}0.0012\\ -0.0007\end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix}-0.0267\\ 2.6723\\ -0.0792\\ 7.9236\end{bmatrix}^T, \quad \boldsymbol{D} = -0.0008$$

$$\boldsymbol{K_k} = [0.0060\ 0.0009\ 0.0242\ -0.1218]^T.$$

The input $\boldsymbol{u_k}$ and the innovation $\boldsymbol{e_k}$ are both random sequences with a Gaussian distribution of unit variance. With these choices, the root mean square amplitude of the deterministic and the stochastic parts of the output have a ratio of around 15dB.

A Monte Carlo simulation with 1000 realizations, each of length $N = 10000$, was performed. For each realization, an ARARMAX model was identified using the parameter choices $\ell = 4$, $\imath = 15$, $n_2 = 2$, and $n_3 = 2$. In each Monte Carlo realization, the same deterministic input $\boldsymbol{u_k}$ was used, but a different innovation $\boldsymbol{e_k}$ was generated. Inspection of the singular values of the Hankel matrices $\boldsymbol{H_{1|\imath}}$, $\boldsymbol{\Lambda_{1|\imath}}$, and $\boldsymbol{G_{1|\imath}}$, reveals that $\boldsymbol{H_{1|\imath}}$ has two significant singular values, $\boldsymbol{\Lambda_{1|\imath}}$ has four, and $\boldsymbol{G_{1|\imath}}$ has four as well, which justifies the choice of an ARARMAX model as well as the choices of n_2 and n_3 for the identification.

The poles estimated in each Monte Carlo run are plotted in fig. 1a for the complete ARARMAX system. They coincide with the poles of the stochastic subsystem. The uncertainty of the poles $-0.9615 \pm 0.1313i$, that are excited by the stochastic subsystem only, is clearly larger than for the other two poles. When the poles of the deterministic subsystem are determined as the eigenvalues of $\boldsymbol{\hat{A}_{22}}$, as in fig. 1b, a loss of accuracy is observed. This is due to the fact that $\boldsymbol{\hat{A}_{23}}$ and $\boldsymbol{\hat{A}_{32}}$ are nonzero for finite $\jmath$, as explained in section 5.6. Only $\boldsymbol{\hat{B}_3}$ is exactly zero for finite $\jmath$.

The zeros of the deterministic subsystem, estimated as the eigenvalues of $\boldsymbol{\hat{A}_{22}} - \boldsymbol{\hat{B}_2}\boldsymbol{\hat{D}}^{-1}\boldsymbol{\hat{C}_2}$, are plotted in fig. 1. Although on average the zeros, estimated in each Monte Carlo run, are close to the true values, they have a large variance error.

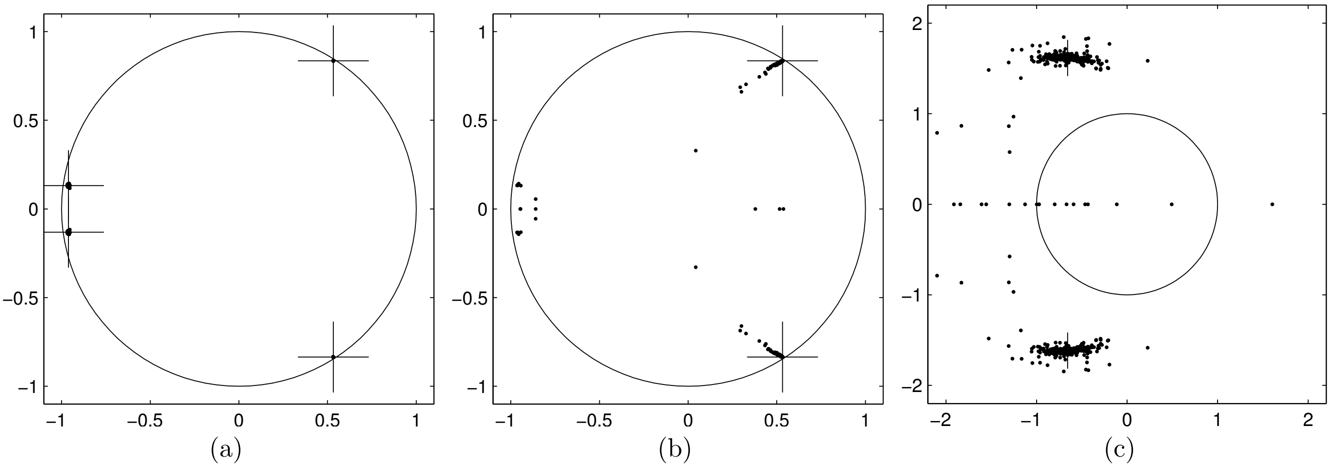

Fig. 1. Simulation example: all poles (a), deterministic poles (b) and deterministic zeros (c), plotted in the complex plane. The true values are indicated with crosses, the values estimated in each of the 1000 Monte Carlo realizations are indicated with dots.

7. CONCLUSIONS

An algorithm was presented for the identification of generalized model structure in state space form. The algorithm is of the subspace type and consists of two steps: 1) the nonparametric identification of the impulse responses of the deterministic subsystem and the output correlations of the stochastic subsystem and 2) the realization of the system matrices. The algorithm was shown to be strongly consistent under classic assumptions. Future research will concentrate on further improving its finite sample behavior. A simulation example was used to illustrate the performance of the method on the identification of an ARARMAX system. Despite the relatively low SNR, the poles were accurately estimated, while the zeros of the identified deterministic subsystem showed a large variance.

The proposed algorithm can be very efficiently implemented using the LQ decomposition, were the computation of the Q factor can be completely avoided. Due to space limitations, implementation details will be given elsewhere.

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